

A VECTORIAL INGHAM-BEURLING THEOREM

ALIA BARHOUMI, VILMOS KOMORNIK, AND MICHEL MEHRENBERGER

ABSTRACT. Baiocchi et al. generalized a few years ago a classical theorem of Ingham and Beurling by means of divided differences. The optimality of their assumption has been proven by the third author of this note. The purpose of this note is to extend these results to vector coefficient sums.

1. INTRODUCTION

Let $\Omega := (\omega_k)_{k \in \mathbb{Z}}$ be a family of real numbers satisfying the gap condition

$$(1.1) \quad \gamma := \inf_{k \neq n} |\omega_k - \omega_n| > 0.$$

Let us denote by $D^+ = D^+(\Omega)$ its Pólya upper density, defined by the formula $D^+ := \lim_{r \rightarrow \infty} r^{-1} n^+(r)$, where $n^+(r)$ denotes the largest number of terms of the sequence $(\omega_k)_{k \in \mathbb{Z}}$ contained in an interval of length r .

Let $(U_k)_{k \in \mathbb{Z}}$ be a corresponding family of unit vectors in some complex Hilbert space H and consider the sums

$$(1.2) \quad x(t) = \sum_{k \in \mathbb{Z}} x_k U_k e^{i\omega_k t}$$

with square summable complex coefficients x_k . We are interested in the validity of the estimates

$$(1.3) \quad \int_I |x(t)|_H^2 dt \asymp \sum_{k \in \mathbb{Z}} |x_k|^2$$

where I is a bounded interval of length denoted by $|I|$ and where we write $A \asymp B$ if there exist two positive constants c_1, c_2 satisfying $c_1 A \leq B \leq c_2 A$.

We have the following result which generalizes a theorem of Ingham [4]:

Theorem 1.1.

- (a) If $|I| > 2\pi D^+$, then the estimates (1.3) hold true.
- (b) If the estimates (1.3) hold true and H has a finite dimension d , then $|I| \geq 2\pi D^+/d$.

For $d = 1$ the theorem reduces to the scalar case due to Beurling [2].

The theorem is sharp in the following sense. Given any real number α between D^+ and D^+/d , there exists a partition $\Omega = \Omega_1 \cup \dots \cup \Omega_d$ of Ω such that $\max_j D^+(\Omega_j) = \alpha$. Fix an orthonormal basis E_1, \dots, E_d of H and set $U_k = E_j$ if $\omega_k \in \Omega_j$. Then using the identity

$$(1.4) \quad \int_I \left\| \sum_{k \in \mathbb{Z}} x_k U_k e^{i\omega_k t} \right\|_H^2 dt = \sum_{j=1}^d \int_I \left| \sum_{\omega_k \in \Omega_j} x_k e^{i\omega_k t} \right|^2 dt$$

and applying the scalar case of the theorem we conclude that the estimates (1.3) hold if $|I| > 2\pi\alpha$, and they do not hold if $|I| < 2\pi\alpha$.

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We prove this theorem in the next section and then we extend the result to the case of a weakened gap condition.

We refer to [5] for many control theoretical applications of theorems of this type.

2. PROOF OF THE THEOREM

Part (a) readily follows from the scalar case. Indeed, fixing an orthonormal basis $(E_n)_{n \in \mathbb{N}}$ of the closed linear hull of $(U_k)_{k \in \mathbb{Z}}$ in H and developing the vectors U_k into Fourier series: $U_k = \sum_{n \in \mathbb{N}} u_{kn} E_n$, for $|I| > 2\pi D^+$ we have

$$\int_I \left\| \sum_{k \in \mathbb{Z}} x_k U_k e^{i\omega_k t} \right\|_H^2 dt = \sum_{n \in \mathbb{N}} \int_I \left| \sum_{k \in \mathbb{Z}} x_k u_{kn} e^{i\omega_k t} \right|^2 dt \asymp \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{Z}} |x_k u_{kn}|^2 = \sum_{k \in \mathbb{Z}} |x_k|^2.$$

For the proof of part (b) we adapt the approach developed in [3] and [6]. We set $\gamma_k := 2\pi|I|^{-1}k$ for brevity. Given three real numbers y, r, R with $r, R > 0$, we introduce the orthogonal projections $P_r : L^2(I, H) \rightarrow V_r$ and $Q_{r+R} : L^2(I, H) \rightarrow W_{r+R}$ onto the finite-dimensional linear subspaces

$$V_r := \text{Vect} \{ U_k e^{i\omega_k t} : |\omega_k - y| < r \}$$

and

$$W_{r+R} := \text{Vect} \{ U e^{i\gamma_n t} : |\gamma_n - y| < r + R \text{ and } U \in H \}.$$

Setting $S := P_r \circ Q_{r+R} \circ i$ where i denotes the injection $i : V_r \hookrightarrow L^2(I, H)$, we obtain a linear map of V_r into itself. We are going to study its trace. We denote by $\Omega_r := \{ \omega_k : |\omega_k - y| < r, k \in \mathbb{Z} \}$ and $\Gamma_{r+R} := \{ \gamma_k : |\gamma_k - y| < r + R, k \in \mathbb{Z} \}$ the sets of exponents figuring in the definition of V_r and W_{r+R} .

Lemma 2.1. *We have*

$$|\text{tr}(S)| \leq d \text{ Card}(\Gamma_{r+R}).$$

Proof. We have

$$\|S\| \leq \|P_r\| \cdot \|Q_{r+R}\| \leq 1.$$

Hence the eigenvalues of S have modulus ≤ 1 and therefore

$$|\text{tr}(S)| \leq \text{rang}(S) \leq \dim(W_{r+R}).$$

Since $\dim(W_{r+R}) = d \text{ Card}(\Gamma_{r+R})$, the lemma follows. \square

Lemma 2.2. *Writing $e_k(t) := U_k e^{i\omega_k t}$ for brevity, we have*

$$\text{tr}(S) = \text{Card}(\Omega_r) + \sum_{|\omega_k - y| < r} ((Q_{r+R} - \text{Id})e_k, P_r \varphi_k)_H$$

where $(\varphi_k)_{k \in \mathbb{Z}}$ is a bounded biorthogonal family to $(e_k)_{k \in \mathbb{Z}}$ in $L^2(I, H)$.

Proof. We have

$$\begin{aligned} \text{tr}(S) &= \sum_{|\omega_k - y| < r} (S e_k, \varphi_k)_{L^2(I, H)} = \sum_{|\omega_k - y| < r} (Q_{r+R} e_k, P_r \varphi_k)_{L^2(I, H)} \\ &= \sum_{|\omega_k - y| < r} (e_k, P_r \varphi_k)_{L^2(I, H)} + \sum_{|\omega_k - y| < r} ((Q_{r+R} - \text{Id})e_k, P_r \varphi_k)_{L^2(I, H)}. \end{aligned}$$

Since $P_r e_k = e_k$, we have $(e_k, P_r \varphi_k)_{L^2(I, H)} = 1$ and the result follows. \square

Lemma 2.3. *We have*

$$\|(Q_{r+R} - \text{Id})e_k\| = O(1/R) \quad (R \rightarrow \infty)$$

uniformly for all $y \in \mathbb{R}$, $r > 0$ and k satisfying $|\omega_k - y| < r$.

Proof. Fixing an orthonormal basis E_1, \dots, E_d of H and setting

$$f_{n,j}(t) := |I|^{-1/2} E_j e^{i\gamma_n t}$$

we have

$$e_k = \sum_{n \in \mathbb{Z}} \sum_{j=1}^d (e_k, f_{n,j})_{L^2(I,H)} f_{n,j}$$

and

$$Q_{r+R} e_k = \sum_{|\gamma_n - y| < r+R} \sum_{j=1}^d (e_k, f_{n,j})_{L^2(I,H)} f_{n,j}.$$

Applying Parseval's equality it follows that

$$\|(Q_{r+R} - \text{Id})e_k\|^2 = \sum_{|\gamma_n - y| \geq r+R} \sum_{j=1}^d |(e_k, f_{n,j})_{L^2(I,H)}|^2.$$

Since

$$(2.1) \quad |(e_k, f_{n,j})_{L^2(I,H)}| = |I|^{-1/2} \left| \int_I (U_k, E_j)_H e^{i(\omega_k - \gamma_n)t} dt \right| \leq \frac{2|I|^{-1/2}}{|\omega_k - \gamma_n|},$$

and $|\omega_k - y| < r$, then we obtain that

$$\begin{aligned} \|(Q_{r+R} - \text{Id})e_k\|^2 &\leq 4d|I|^{-1} \sum_{|\gamma_n - y| \geq r+R} \frac{1}{|\omega_k - \gamma_n|^2} \\ &\leq 4d|I|^{-1} \sum_{|\gamma_n - y| > r+R} \frac{1}{||y - \gamma_n| - r|^2} \\ &\leq 8d|I|^{-1} \sum_{n=0}^{\infty} \frac{1}{|2\pi|I|^{-1}n + R|^2}. \end{aligned}$$

Since the last expression doesn't depend on r , y and is $O(1/R)$ as $R \rightarrow \infty$, the lemma follows. \square

Now the proof of part (b) of Theorem 1.1 can be completed as follows. By the above lemmas we have

$$\begin{aligned} d \text{Card}(\Gamma_{r+R}) &\geq |\text{tr}(S)| = \left| \text{Card}(\Omega_r) + \sum_{|\omega_k - y| < r} ((Q_{r+R} - \text{Id})e_k, P_r \varphi_k)_H \right| \\ &\geq \text{Card}(\Omega_r) - O(1/R) \text{Card}(\Gamma_{r+R}) \end{aligned}$$

and therefore

$$\text{Card}(\Omega_r) \leq (d + O(1/R)) \text{Card}(\Gamma_{r+R}), \quad R \rightarrow \infty.$$

Hence

$$D^+ = \lim_{r \rightarrow \infty} \frac{\text{Card}(\Omega_r)}{r} \leq \lim_{R \rightarrow \infty} \lim_{r \rightarrow \infty} (d + O(1/R)) \frac{\text{Card}(\Gamma_{r+R})}{r+R} \cdot \frac{r+R}{r} = \lim_{R \rightarrow \infty} (d + O(1/R)) \frac{|I|}{2\pi} = \frac{d|I|}{2\pi}$$

and therefore $|I| \geq 2\pi D^+/d$ as claimed.

3. THE CASE OF THE DIVIDED DIFFERENCES

The gap condition (1.1) of the theorem may be weakened. Following [1] let $(\omega_k)_{k \in \mathbb{Z}}$ be a nondecreasing sequence of real numbers satisfying for some positive integer M and for some positive real number γ' the weakened gap condition

$$(3.1) \quad \omega_{k+M} - \omega_k \geq M\gamma' \quad \text{for all } k \in \mathbb{Z}.$$

This implies that $D^+ < \infty$. For $j = 1, \dots, M$ and $m \in \mathbb{Z}$ we say that $\omega_m, \dots, \omega_{m+j-1}$ forms a γ' -close exponent chain if

$$\begin{cases} \omega_m - \omega_{m-1} \geq \gamma', \\ \omega_k - \omega_{k-1} < \gamma' \quad \text{for } k = m+1, \dots, m+j-1, \\ \omega_{m+j} - \omega_{m+j-1} \geq \gamma'. \end{cases}$$

Then we define the divided differences $f_\ell = [\omega_m, \dots, \omega_\ell]$ for $\ell = m, \dots, m+j-1$, defined by the formula

$$[\omega_m, \dots, \omega_\ell](t) := (it)^{\ell-1} \int_0^1 \int_0^{s_m} \dots \int_0^{s_{\ell-2}} \exp(i[s_{\ell-1}(\omega_\ell - \omega_{\ell-1}) + \dots + s_m(\omega_{m+1} - \omega_m) + \omega_m]) t \, ds_{\ell-1} \dots ds_m.$$

We can now state a generalization of Theorem 1.1:

Theorem 3.1. *Theorem 1.1 holds true if (1.1) is replaced by (3.1) and $e^{i\omega_k t}$ is replaced by $f_k(t)$.*

Proof. Most of the proof of Theorem 1.1 may be easily adapted. For part (b) we have to replace the estimate (2.1) by the following:

$$(3.2) \quad \left| \int_I (U_k, E_j)_H f_k(t) e^{-i\gamma_n t} dt \right| \leq \left| \int_I f_k(t) e^{-i\gamma_n t} dt \right| \leq \frac{C}{|\omega_k - \gamma_n|},$$

with a constant C depending only on γ' , M and I . This is shown by arguing similarly as in [6]. We have

$$A := \int_I f_k(t) e^{-i\gamma_n t} dt = \int_I g(t) e^{i\omega_k t} e^{-i\gamma_n t} dt$$

with

$$g(t) = [\omega_m - \omega_k, \dots, \omega_k - \omega_k](t).$$

Integrating by parts in $I = (a, b)$ we obtain that

$$A = \left[\frac{1}{i\omega_k - i\gamma_n} g(t) e^{i\omega_k t} e^{-i\gamma_n t} \right]_a^b - \int_I \frac{1}{i\omega_k - i\gamma_n} g'(t) e^{i\omega_k t} e^{-i\gamma_n t} dt.$$

Now a direct computation shows that for any real numbers μ_1, \dots, μ_r the divided differences satisfy the inequality

$$[\mu_1, \dots, \mu_r]'(t) \leq \frac{(r-1)t^{r-2}}{(r-1)!} + (|\mu_r - \mu_{r-1}| + \dots + |\mu_2 - \mu_1| + |\mu_1|) \frac{t^{r-1}}{(r-1)!}.$$

Thus, in our case, thanks to the γ' -close exponent property, we have

$$|g'(t)| \leq (k-m) \frac{t^{k-m-1}}{(k-m)!} + (k-m)\gamma' \frac{t^{k-m}}{(k-m)!}$$

and this yields (3.2). \square

REFERENCES

- [1] C. Baiocchi, V. Komornik, P. Loreti, *Ingham-Beurling type theorems with weakened gap conditions*, Acta Math. Hungar. 97 (1–2) (2002), 55–95.
- [2] J.N.J.W.L. Carleson, P. Malliavin (editors), *The Collected Works of Arne Beurling*, Volume 2, Birkhäuser, 1989.
- [3] K. Gröchenig, H. Razafinjato, *On Landau's necessary conditions for sampling and interpolation of band-limited functions*, J. London Math. Soc. (2), 54 (1996), 557–565.
- [4] A. E. Ingham, *Some trigonometrical inequalities with applications in the theory of series*, Math. Z. 41 (1936), 367–379.
- [5] V. Komornik, P. Loreti, *Fourier Series in Control Theory*, Springer-Verlag, New York, 2005.
- [6] M. Mehrenberger, *Critical length for a Beurling type theorem*, Bol. Un. Mat. Ital. B (8), 8-B (2005), 251–258.

DÉPARTEMENT DE MATHÉMATIQUE, FACULTÉ DES SCIENCES DE MONASTIR,, 5019, MONASTIR, TUNISIE

E-mail address: `Alia.Barhoumi@isimm.rnu.tn`

DÉPARTEMENT DE MATHÉMATIQUE, UNIVERSITÉ LOUIS PASTEUR, 7 RUE RENÉ DESCARTES, 67084 STRASBOURG CEDEX, FRANCE

E-mail address: `komornik@math.u-strasbg.fr`

DÉPARTEMENT DE MATHÉMATIQUE, UNIVERSITÉ LOUIS PASTEUR, 7 RUE RENÉ DESCARTES, 67084 STRASBOURG CEDEX, FRANCE

E-mail address: `mehrenbe@math.u-strasbg.fr`